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On the behaviour of small disturbances in plane Couette flow. Part 3. The phenomenon of mode-pairing

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In two earlier papers (Gallagher & Mercer 1962, 1964) the results for the first four eigenvalues of the problem of the stability of plane Couette flow were given. The first twelve eigenvalues have been calculated by the same method and the results show that the manner in which the eigenvalues join to form complex pairs depends on α, the wavenumber of the disturbance. They also would appear to indicate that mode-crossing always occurs.

1. Introduction

In Gallagher & Mercer (1962) the results for the first eigenvalue of the problem of infinitesimal disturbances to plane Couette flow of a viscous and incompressible fluid were given. In a subsequent paper (Gallagher & Mercer 1964) the results for the first four eigenvalues showed excellent agreement with the results of Southwell & Chitty (1930) whereas a discrepancy with those of Grohne (1954) was found. In view of recent comments by Davey (1973) and correspondence with Reid concerning his recent work (1974) with regard to mode-crossing, it was thought desirable to present the results for the first twelve eigenvalues for various values of α. These results were recomputed on a faster and larger computer, the ICL 1907, thus enabling larger matrices and many more cases to be considered than was feasible on the DEUCE, the machine used previously. The reader is referred to the earlier paper for details of the method used.

The eigenvalues ξ are those of the Orr–Sommerfeld equation

$$v^{\text{iv}} - 2\alpha^2 v'' + \alpha^4 v - i\alpha R(y-\xi)(v''-\alpha^2 v) = 0$$

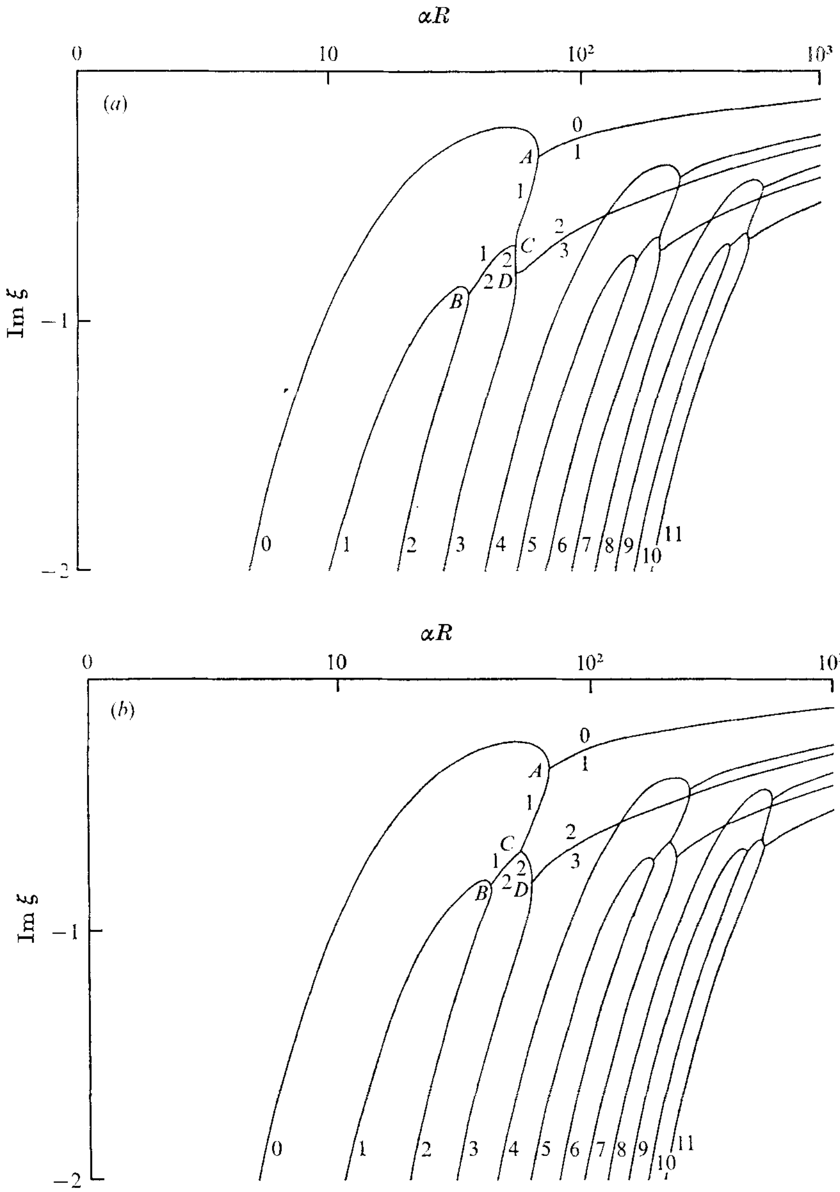
with

$$v(\pm 1) = v'(\pm 1) = 0,$$

which arises from considering infinitesimal disturbances of the form

$$v(y)\exp i\alpha(x-\xi t),$$

R being the Reynolds number. Davey (1973) stated that the previous work of Gallagher & Mercer (1964) showed that mode-crossing did not occur. While this is true for the limited set of results presented in that paper for comparison with those of Grohne, namely the first four eigenvalues for $\alpha = 1$ and $0 \leqslant \alpha R \leqslant 1000$, it is not true in general. Indeed mode-crossing appears in the higher eigenvalues for this case and for all values of α examined and it is conjectured that mode-crossing is always present.

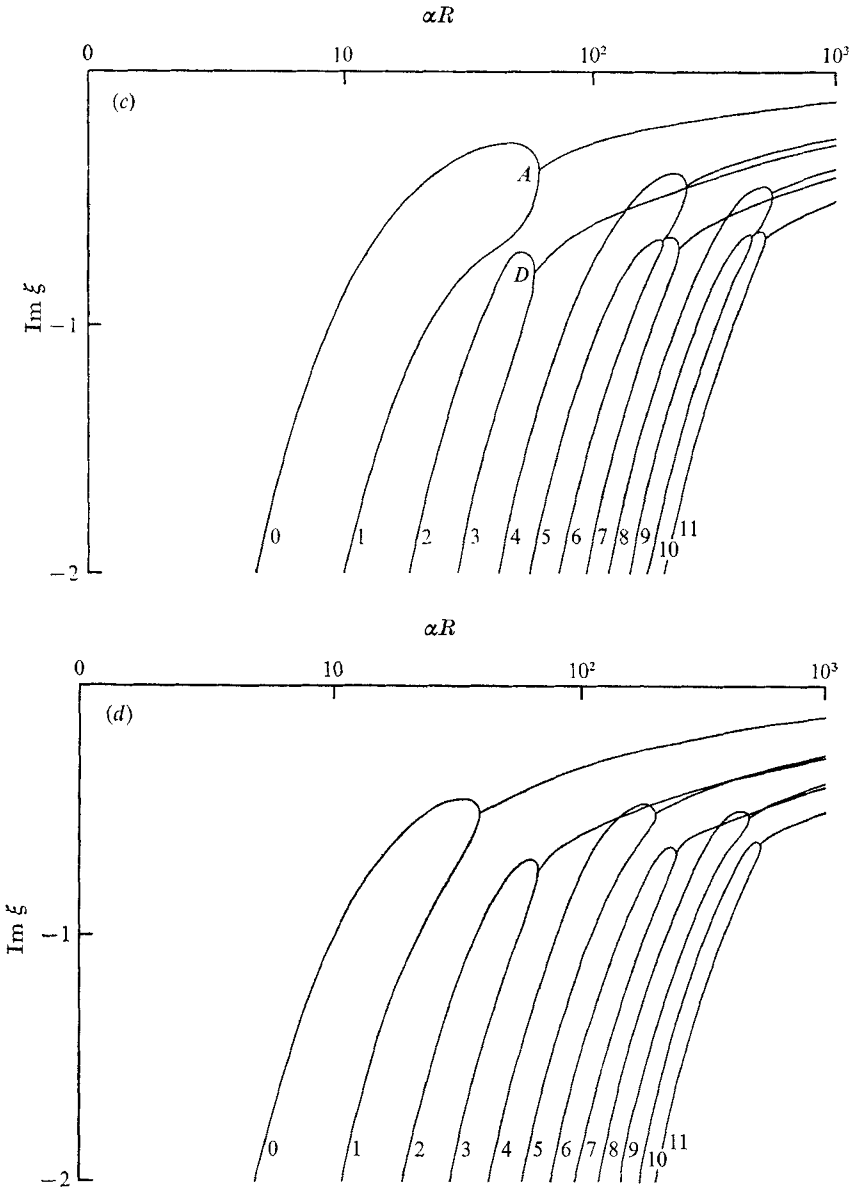


FIGURES 1 (a, b). For legend see facing page.

2. Results

Figures 1 (a)–(d) show the results for $\operatorname{Im} \xi$ plotted against αR for the first twelve eigenvalues for the cases $\alpha = 0$, 0·6, 1 and 2 respectively. These have been labelled 0, 1, 2, ..., 11 using Grohne's convention. In the following, 'pairing' means that two eigenvalues join to form a complex pair of the form $\pm \operatorname{Re} \xi + i \operatorname{Im} \xi$.

In figure 1 (a), the case $\alpha = 0$, it will be seen that ξ_1 and ξ_2 join to form a complex pair at B, being purely imaginary for smaller values of αR. At C they become



FIGURES 1(*c*, *d*). Im ξ *vs*. αR. (*a*) $\alpha = 0$. (*b*) $\alpha = 0{\cdot}6$. (*c*) $\alpha = 1$. (*d*) $\alpha = 2$.

purely imaginary again, with ξ_1 pairing with ξ_0 at A and ξ_2 pairing with ξ_3 at D. The next four eigenvalues pair in a similar manner as do the last four.

As α is increased the portion BC of the curve gradually shrinks (see figure 1 (*b*) for $\alpha = 0{\cdot}6$) until eventually at $\alpha = 0{\cdot}9$ (more precisely at some value of α between 0·91 and 0·92) ξ_1 and ξ_2 cease to pair and the first four eigenvalues adopt a new pattern of modal pairing. This is illustrated in figure 1 (*c*) for $\alpha = 1$. Note, however, that the subsequent eigenvalues are still pairing as in figure 1 (*a*).

αR	1000	1500	2000	2500	3000	Asymptotic values
$-(\alpha R)^{\frac{1}{3}}\,\mathrm{Im}\,\xi_{0,1}$	1·13	1·12	1·11	1·11	1·11	1·06
$-(\alpha R)^{\frac{1}{3}}\,\mathrm{Im}\,\xi_{4,5}$	2·57	2·56	2·56	2·55	2·55	2·52
$-(\alpha R)^{\frac{1}{3}}\,\mathrm{Im}\,\xi_{2,3}$	3·01	3·01	3·01	3·01	3·01	3·04
$-(\alpha R)^{\frac{1}{3}}\,\mathrm{Im}\,\xi_{6,7}$	4·24	4·24	4·24	4·24	4·24	4·31

TABLE 1

Eventually as α is increased still further all twelve eigenvalues adopt the new pattern of modal pairing as illustrated in figure 1 (*d*) for the case $\alpha = 2$.

It may be conjectured that, for any value of $\alpha > 0{\cdot}9$ approximately, a finite number of eigenvalues will pair in the pattern of figure 1 (*d*) while the remainder will pair in the pattern of figure 1 (*a*). It would also appear likely from the present results that mode-crossing always occurs.

A comparison with Reid's (1974) results for the first eight eigenvalues for $\alpha = 0$ showed good agreement except for small values of αR in the odd modes. This is attributed by him to a lack of uniformity as $\alpha R \to 0$ in his asymptotic approximations. The asymptotic values for $\mathrm{Im}\,\xi$ as $\alpha R \to \infty$ are given by him for this case to be of the form $C(\alpha R)^{-\frac{1}{3}}$, where C is a constant depending on the mode. These are compared with the actual values obtained for various values of αR in table 1. These were calculated using matrices of order 30, the maximum size of matrix used. The system of labelling the eigenvalues differs somewhat from that adopted by Reid. The present system was chosen so as to remain invariant with respect to α.

The present calculations were also compared with some of Davey's unpublished results for the case $\alpha = 0$ and excellent agreement was obtained.

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